# Time-Discretization of Nonlinear Systems with Delayed Multi-Input Using Taylor Series

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This study proposes a new scheme for the sampled-data representation of nonlinear systems with time-delayed multi-input. The proposed scheme is based on the Taylor-series expansion and zero-order hold assumption. The mathematical structure of a new discretization scheme is explored. On the basis of this structure, the sampled-data representation of nonlinear systems including time-delay is derived. The new scheme is applied to nonlinear systems with two inputs and then the delayed multi-input general equation is derived. The resulting time-discretization provides a finite-dimensional representation of nonlinear control systems with time-delay enabling existing controller design techniques to be applied to them. In order to evaluate the tracking performance of the proposed scheme, an algorithm is tested for some of the examples including maneuvering of an automobile and a 2-DOF mechanical system.

**Key Words :** Nonlinear System, Time-Discretization, Time-Delay, Multi-Input, Taylor-Series

## 1. Introduction

Resolving time-delay in control systems will become increasingly important in the near future as Internet technology further develops and evolves. There are two reasons why time-delay is received special attention in the field of control systems. First, time-delay is increasing due to the communication needs and complex computations involved in control systems. Digital controllers using communications and having increased computational requirements induce such a time-delay. In embedded control systems, the effects of this time-delay cannot be ignored due to the communication and increased computation. Second, control systems with time-delay exhibit complex behavior due to infinite dimensionality, even in the case of linear system with a constant time-delay in the input or states has infinite dimensionality when expressed in the continuous time domain. Therefore, the controller design technique developed in the finite-dimensional systems during the last few decades cannot be applied to systems having any time-delay in the variables. Thus, it is necessary to develop a control system design method that resolves this time-delay.

The engineering literature dealing with time-delayed system and a discretization is very extensive. Most of this literature deals with linear time-delay control systems and, in particular,

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with the stability and robustness related to time-delay. In Choi et al. (1999), the authors proposed a new control scheme applicable to systems with time-delay, which is based on the conventional position-position feedback-type controller. The stability of this control system has been proved using scattering theory and compared to the conventional ones. Jeong and Lee (1995) proposed a method of designing a robust time-delayed teleoperator robot system based on optimization. The proposed teleoperator control system deals with the robustness of teleoperation, especially, during the contact phase.

Time Delay Control (TDC) utilizing the estimated uncertainties of general nonlinear systems using the time-delay method is actively studied. Choi and Baek (2002) studied magnetic levitation systems required to have a large operating range in many applications. TDC was applied to a single-axis magnetic levitation system and a reduced-order observer was utilized to estimate states excluding measurable states in the control law. Lee and Chang (1999) studied the input/output linearization (IOL) method using TDC and a time-delay observer. This method enables the IOL method to be applied to plants even when not all of the states of the plant are measurable or the measured plant output is very noisy. In the study by Byeon and Song (1997), a position control system was developed for the throttle actuator system that uses one throttle actuation to obtain low volume and a DC servo motor for fast response. In order to drive the DC motor, the PWM signal generator and PWM amplifier were built and interfaced to the motor and controller. Also, the time-delay control (TDC) law was used as a basic control algorithm. A method of varying the reference model of the TDC with respect to the degree of change in the target throttle angle was proposed in this study. To apply TDC to a real system, Kwon et al. (2002) designed a Time Delay Controller to guarantee stability. In a previous study the sufficient stability condition of the TDC was described for general plants. A new sufficient stability condition for TDC of general plants with finite time-delay is proposed.

Hong and Wu (1994) derive sufficient conditions for the zeros of the polynomial to be either inside the unit disk in the complex plane or at least one zero not inside the unit disk by examining the coefficients of a given polynomial in the linear discrete system. Kang and Park (1999) experimentally confirm the fundamental dynamic properties of an electrodynamic structure. The discretization effects are examined for the conversion of continuous properties such as mass, stiffness, and surface charge into discrete quantities. In the systems considered, the linearized characteristics are well-matched with the nonlinear systems in the sense that the linearized effects predominate over the high-order nonlinear terms.

In the field of a discretization, conventional numerical techniques such as the Euler and Runge-Kutta method have been used for obtaining the sampled-data representation for the original continuous-time system (Franklin et al., 1998), which does not have delay. All of these approaches require a small time step in order to be deemed accurate, and this may not be the case in control applications where slow sampling and large sampling periods are inevitably introduced due to physical and technical limitations (Kazantzis and Kravaris, 1997; 1999; Vaccaro, 1995). Another interesting result for the discretization of the delay-free nonlinear system can be found in the Carleman linearization method (Svoronos et al., 1994). However, this method is useful only for the low-dimensional system since its dimensionality increases rapidly with the continuous model's dimension and the degree of the desired accuracy.

In general, most, if not all, industrial controllers are currently implemented digitally. In the design of model-based digital controllers, for systems of both process and non-process type, two general approaches are available. First, a continuous time controller is designed based on a continuous time system model, followed by a digital redesign of the controller in the discrete-time domain in order to approximate the performance of the original continuous-time controller. Second, a direct digital design approach can be followed based on a discrete-time model (sampled-data representation) of the system, in which the controller is directly designed in the discrete-time domain. It is apparent that this alternative approach is more attractive when dealing directly with the issue of sampling. Indeed, the effect of sampling on the system-theoretic properties of the continuous-time system is very important because these properties are associated with the attainment of the design objectives. It should be emphasized that in both design approaches, time discretization of either the controller or the system model is necessary. Furthermore, notice that in the controller design for time-delay systems, the first approach is troublesome due to the infinite-dimensional nature of the underlying system dynamics. As a result, the second approach becomes more desirable and will be pursued in the present study.

This paper expands the well-known time-discretization of the linear time-delay system (Franklin et al., 1998 ; Vaccaro, 1995) to nonlinear control systems with delayed multi-input. The proposed discretization scheme applies the Taylor series expansion according to the mathematical structure developed for the delay-free nonlinear system (Kazantzis and Kravaris, 1997 ; 1999) and delayed single-input nonlinear system (Kazantzis et al., 2003).

## 2. Nonlinear System with Time-Delay Single-Input

Single-input nonlinear continuous-time control systems can be expressed with the following state-space representation :

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t-D) \tag{1}$$

where $x \in X \subset R^n$ is the vector of states and $X$ is an open and connected set, $u \in R$ is the input variable and $D$ is a constant time-delay. It is assumed that $f(x)$, $g(x)$ are real analytic vector fields on $X$.

An equidistant grid on the time axis with mesh $T=t_{k+1}-t_k>0$ is considered, where $[t_k, t_{k+1})=[kT, (k+1)T)$ is the sampling interval and $T$ is the sampling period. It is assumed that system Eq. (1) is driven by an input that is piecewise constant over the sampling interval, i.e. the zero-order hold (ZOH) assumption holds true :

$$u(t)=u(kT)\equiv u(k)=constant \tag{2}$$

for $kT \leq t < kT+T$. Furthermore, let :

$$D=qT+\gamma \tag{3}$$

where $q \in \{0, 1, 2, \cdots\}$ and $0<\gamma\leq T$. Equivalently, the time-delay $D$ is customarily represented as an integer multiple of the sampling period plus a fractional part of $T$ (Franklin et al., 1998 ; Vaccaro, 1995). Under the ZOH assumption and using the above notation, it is rather straightforward to verify that the delayed input variable attains the following two distinct values within the sampling interval (Vaccaro, 1995):

$$u(t-D)=\begin{cases} u(kT-qT-T)\equiv u(k-q-1) & \text{if } kT\leq T<kT+\gamma \\ u(kT-qT)\equiv u(k-q) & \text{if } kT+\gamma\leq t<kT+T \end{cases} \tag{4}$$

Under the above preliminaries, the mathematical expression of the time-discretization of a single-input nonlinear system with time-delay will be presented.

Initially, delay-free ($D=0$) nonlinear control systems are considered with a state space representation of the form :

$$\frac{dx(t)}{dt}=f(x(t))+g(x(t))u(t) \tag{5}$$

Under the ZOH assumption and within the sampling interval, the solution of Eq. (5) is expanded in a uniformly convergent Taylor series (Vidyasagar, 1978) and the resulting coefficients can be easily computed by taking successive partial derivatives of the right-hand-side of Eq. (5):

$$\begin{aligned} x(k+1) &= x(k)+\sum_{l=1}^{\infty}\frac{T^l}{l!}\frac{d^l x}{dt^l}\bigg|_{t_k} \\ &= x(k)+\sum_{l=1}^{\infty}A^{[l]}(x(k), u(k))\frac{T^l}{l!} \end{aligned} \tag{6}$$

where $x(k)$ is the value of the state vector $x$ at time $t=t_k=kT$ and $A^{[l]}(x, u)$ are determined recursively by :

$$\begin{aligned} A^{[1]}(x, u) &= f(x)+ug(x) \\ A^{[l+1]}(x, u) &= \frac{\partial A^{[l]}(x, u)}{\partial x}(f(x)+ug(x)) \end{aligned} \tag{7}$$

with $l=1, 2, 3, \cdots$.

The sampled-data representation of the nonlinear system with delayed single-input can be derived from Eq. (6) resulting in Eq. (8) (Kazantzis et al., 2003).

$$\begin{aligned} x(kT+\gamma) &= x(kT) + \sum_{l=1}^{\infty} A^l(x(kT), u(k-q-1))\frac{\gamma^l}{l!} \\ &\qquad \text{if } kT \le t < kT+\gamma \\ x(kT+T) &= x(kT+\gamma) + \sum_{l=1}^{\infty} A^l(x(kT+\gamma), u(k-q))\frac{(T-\gamma)^l}{l!} \\ &\qquad \text{if } kT+\gamma \le t < kT+T \end{aligned} \tag{8}$$

where $x(k)$ and $A^{[l]}(x, u)$ are the same as in the above delay-free case.

As a result, the time-discretization of a nonlinear control system with delayed input is computed by

$$\begin{aligned} x(k+1) &= x(k) + \sum_{l=1}^{\infty} A^l(x(k), u(k-q-1))\frac{\gamma^l}{l!} \\ &+ \sum_{l=1}^{\infty} A^l\left(\left(x(k) + \sum_{i=1}^{\infty} A^i(x(k), u(k-q-1))\frac{\gamma^i}{i!}\right), u(k-q)\right)\frac{(T-\gamma)^l}{l!} \end{aligned} \tag{9}$$

## 3. Nonlinear System with Time-Delay Multi-Input

As shown in (Kazantzis et al., 2003), the time-discretization of a nonlinear system with single-input time-delay can be obtained using the Taylor series, similarly the expansion of the single input system to multi-input system is possible. Input delays that are either within one sampling period or larger than one sampling period will be considered in this section.

For simplicity, a system with two inputs will be considered in this section. A two-input nonlinear continuous-time control system can be expressed in the following state-space form :

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t)) + u_1(t-D_1)g_1(x(t)) \\ &+ u_2(t-D_2)g_2(x(t)) \end{aligned} \tag{10}$$

The delays of the inputs are described in Eq. (11), which is derived from Eq. (3),

$$\begin{aligned} u_1(t-D_1) &\dashrightarrow (D_1 = q_1T + \gamma_1) \\ u_2(t-D_2) &\dashrightarrow (D_2 = q_2T + \gamma_2) \end{aligned} \tag{11}$$

If the difference of the delays is less than one sampling period, the inputs are as follows :

$$\begin{aligned} u_1(t-D_1) &= \begin{cases} \text{if } kT \le t < kT+\gamma_1 \\ \quad u_1(kT-q_1T-T) \equiv u_1(k-q_1-1) \\ \text{if } kT+\gamma_1 \le t < kT+T \\ \quad u_1(kT-q_1T) \equiv u_1(k-q_1) \end{cases} \\ u_2(t-D_2) &= \begin{cases} \text{if } kT \le t < kT+\gamma_2 \\ \quad u_2(kT-(q_1+n)T)-T) \equiv u_2(k-(q_1+n)-1) \\ \text{if } kT+\gamma_2 \le t < kT+T \\ \quad u_2(kT-(q_1+n)T) \equiv u_2(k-(q_1+n)) \end{cases} \end{aligned} \tag{12}$$

It is necessary to specify how the sampled-data representation is affected when the multi-inputs with delay are applied to the system. Delays within one sampling period, as well as delays that are larger than one sampling period, will be considered in the following section. This is a necessary procedure that derives a general equation of time-discretization for nonlinear system with time-delay.

### 3.1 The case when difference of delay is less than one sampling period

In this section, the difference of input delays are less than one sampling period is considered. The delayed inputs are depicted in Fig. 1. The horizontal axis indicates time and $u_i(k-D)$ refers to the input of system. In the interval between $k$ and $k+1$, $u_1$ is $u_1(k-q_1-1)$ before time reaches $k+\gamma_1$ and $u_1(k-q_1)$ after $k+\gamma_1$. In a similar manner, $u_2$ will be $u_2(k-q_2-1)$ before time reaches $k+\gamma_2$ and $u_2(k-q_2)$ after $k+\gamma_2$. Thus, the input values are decided depending on the time. Since the difference in the input delays is less than one sampling period, all of the inputs

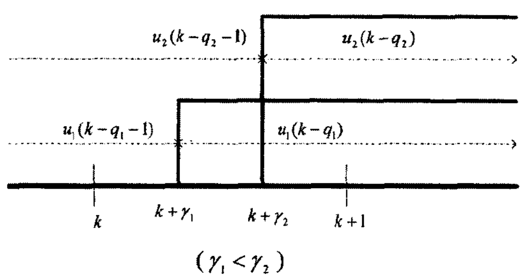


**Fig. 1** Input signal for each time interval in one sampling period

are located in the same sampling period. It should be noted that two cases exist in the input delays such that $\gamma_1 \leq \gamma_2$, $\gamma_2 < \gamma_1$. The values of $q_1$ and $q_2$ are zeros since the delays are less than one sampling period. There are three time intervals in one sampling period such that $kT \leq t < kT + \gamma_1$, $kT + \gamma_1 \leq t < kT + \gamma_2$, and $kT + \gamma_2 \leq t < kT + T$.

The inputs and the corresponding state values can be obtained as follows.

1) Case 1, $\gamma_1 \leq \gamma_2$

If time $t$ is in $kT \leq t < kT + \gamma_1$ as shown in Fig. 1, the inputs and state values can be written as follows:

$$\begin{aligned} &u_1(t-D_1) = u_1(k-q_1-1),\ u_2(t-D_2) = u_2(k-q_2-1) \\ &x(kT+\gamma_1) = x(kT) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_1(k-q_1-1),\ u_2(k-q_2-1)) \frac{\gamma_1^l}{l!} \end{aligned} \tag{13}$$

The input values in the interval $kT \leq t < kT + \gamma_1$ are determined by the input produced by one sampling period ahead.

In the second interval $kT + \gamma_1 \leq t < kT + \gamma_2$, the input and the state values are

$$\begin{aligned} &u_1(t-D_1) = u_1(k-q_1),\ u_2(t-D_2) = u_2(k-q_2-1) \\ &x(kT+\gamma_2) = x(kT+\gamma_1) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_1),\ u_1(k-q_1),\ u_2(k-q_2-1)) \frac{(\gamma_2-\gamma_1)^l}{l!} \end{aligned} \tag{14}$$

In the third interval when time $t$ is in $kT + \gamma_2 \leq t < kT + T$, the input and the state values are as follows:

$$\begin{aligned} &u_1(t-D_1) = u_1(k-q_1),\ u_2(t-D_2) = u_2(k-q_2) \\ &x(kT+T) = x(kT+\gamma_2) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_2),\ u_1(k-q_1),\ u_2(k-q_2)) \frac{(T-\gamma_2)^l}{l!} \end{aligned} \tag{15}$$

2) Case 2, $\gamma_2 < \gamma_1$

In this case, the delay of $u_2$ is larger than the delay of $u_1$, thus the locations of $\gamma_1$ and $\gamma_2$ are reversed. The input and the state values are obtained in a similar way to that in the case 1. In the first interval, $kT \leq t < kT + \gamma_2$, the inputs and the state values can be written as follows:

$$\begin{aligned} &u_2(t-D_2) = u_2(k-q_2-1),\ u_1(t-D_1) = u_1(k-q_1-1) \\ &x(kT+\gamma_2) = x(kT) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_2(k-q_2-1),\ u_1(k-q_1-1)) \frac{(\gamma_2)^l}{l!} \end{aligned} \tag{16}$$

Similarly, for the second interval $kT + \gamma_2 \leq t < kT + \gamma_1$, the input and the state values are as follows:

$$\begin{aligned} &u_2(t-D_2) = u_2(k-q_2),\ u_1(t-D_1) = u_1(k-q_1-1) \\ &x(kT+\gamma_1) = x(kT+\gamma_2) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_2),\ u_2(k-q_2),\ u_1(k-q_1-1)) \frac{(\gamma_1-\gamma_2)^l}{l!} \end{aligned} \tag{17}$$

For the third interval, $kT + \gamma_1 \leq t < kT + T$, the input and the state values are as follows:

$$\begin{aligned} &u_2(t-D_2) = u_2(k-q_2),\ u_1(t-D_1) = u_1(k-q_1) \\ &x(kT+T) = x(kT+\gamma_1) \\ &\quad + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_1),\ u_2(k-q_2),\ u_1(k-q_1)) \frac{(T-\gamma_1)^l}{l!} \end{aligned} \tag{18}$$

The number of intervals in one sampling period is related to the number of inputs. If there is one input, then there are two intervals in one sampling period, and for two inputs three intervals should be considered.

### 3.2 The case when difference of delay is between one and two sampling periods

This section discusses a difference of input delays greater than one sampling period and less than two sampling periods, using a similar approach to that taken in the previous section. Since the difference in input delay is greater than one sampling period, two sampling periods are considered to obtain the inputs and the state values. Thus, six intervals are considered since there are two inputs and two sampling periods as shown in Fig. 2. This figure depicts the inputs of the system in a fashion similar to Fig. 1. The horizontal axis indicates time and $u_i(k-D)$ refers to the input of a system. In the interval between $k$ and $k+2$, $u_1$ is $u_1(k-q_1-1)$ when time is in between $k$ and $k+\gamma_1$, $u_1(k-q_1)$ when time is in between $k+\gamma_1$ and $k+1+\gamma_1$ and $u_1(k-q_1+1)$ when time is after $k+1+\gamma_1$. The input values of $u_2$ will be decided similarly as in $u_1$.

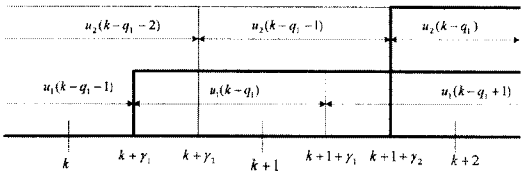


**Fig. 2** Input signal for each time interval in two sampling periods

All of the inputs and the state values are as follows :

If $kT \le t < kT + \gamma_1$, $(q_2 = q_1 + 1)$,
then

$$u_1 = u_1(k-q_1-1),\ u_2 = u_2(k-q_1-2),$$
$$x(kT+\gamma_1) = x(kT) + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_1(k-q_1-1),\ u_2(k-q_1-2))\frac{(\gamma_1)^l}{l!} \tag{19}$$

If $kT + \gamma_1 < t < kT + \gamma_2$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-2),$$
$$x(kT+\gamma_2) = x(kT+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_1),\ u_1(k-q_1),\ u_2(k-q_1-2))\frac{(\gamma_2-\gamma_1)^l}{l!} \tag{20}$$

If $kT + \gamma_2 \le t < kT + T$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-1)$$
$$x(kT+T) = x(kT+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_2),\ u_1(k-q_1),\ u_2(k-q_1-1))\frac{(T-\gamma_2)^l}{l!} \tag{21}$$

If $kT + T \le t < kT + T + \gamma_1$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-1)$$
$$x(kT+T+\gamma_1) = x(kT+T) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T),\ u_1(k-q_1),\ u_2(k-q_1-1))\frac{(\gamma_1)^l}{l!} \tag{22}$$

If $kT + T + \gamma_1 \le t < kT + T + \gamma_2$,
then

$$u_1 = u_1(k-q_1+1),\ u_2 = u_2(k-q_1-1)$$
$$x(kT+T+\gamma_2) = x(kT+T+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T+\gamma_1),\ u_1(k-q_1+1),\ u_2(k-q_1-1))\frac{(\gamma_2-\gamma_1)^l}{l!} \tag{23}$$

If $kT + T + \gamma_2 \le t < kT + 2T$,
then

$$u_1 = u_1(k-q_1+1),\ u_2 = u_2(k-q_1)$$
$$x(kT+2T) = x(kT+T+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T+\gamma_2),\ u_1(k-q_1+1),\ u_2(k-q_1))\frac{(T-\gamma_2)^l}{l!} \tag{24}$$

### 3.3 The case when difference of delay is greater than two sampling periods

A difference of input delays greater than two sampling periods will be considered in this section. In this case, nine intervals are considered since there are two inputs in the system and the difference in the input delays is two sampling periods. Figure 3 shows a diagram of the input signal for each time interval in three sampling periods. The input and state values are obtained in a similar way to that used in Sec. 3.1 and Sec. 3.2. Also, it is assumed that it is a fixed integer.

All of the inputs and state values are as follows.

If $kT \le t < kT + \gamma_1$, $(q_2 = q_1 + 2)$,
then

$$u_1 = u_1(k-q_1-1),\ u_2 = u_2(k-q_1-3)$$
$$x(kT+\gamma_1) = x(kT) + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_1(k-q_1-1),\ u_2(k-q_1-3))\frac{(\gamma_1)^l}{l!} \tag{25}$$

If $kT + \gamma_1 \le t < kT + \gamma_2$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-3)$$
$$x(kT+\gamma_2) = x(kT+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_1),\ u_1(k-q_1),\ u_2(k-q_1-3))\frac{(\gamma_2-\gamma_1)^l}{l!} \tag{26}$$

If $kT + \gamma_2 \le t < kT + T$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-2)$$
$$x(kT+T) = x(kT+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_2),\ u_1(k-q_1),\ u_2(k-q_1-2))\frac{(T-\gamma_2)^l}{l!} \tag{27}$$

If $kT + T \le t < kT + T + \gamma_1$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-q_1-2)$$
$$x(kT+T+\gamma_1) = x(kT+T) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T),\ u_1(k-q_1),\ u_2(k-q_1-2))\frac{(\gamma_1)^l}{l!} \tag{28}$$

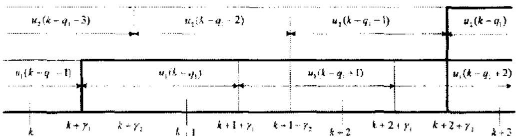

**Fig. 3** Input signal for each time interval in three sampling periods

If $kT+T+\gamma_1 \le t < kT+T+\gamma_2$,
then

$$u_1 = u_1(k-q_1+1),\ u_2 = u_2(k-q_1-2)$$

$$x(kT+T+\gamma_2) = x(kT+T+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T+\gamma_1), u_1(k-q_1+1), u_2(k-q_1-2)) \frac{(\gamma_2-\gamma_1)^l}{l!} \tag{29}$$

If $kT+T+\gamma_2 \le t < kT+2T$,
then

$$u_1 = u_1(k-q_1+1),\ u_2 = u_2(k-q_1-1)$$

$$x(kT+2T) = x(kT+T+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+T+\gamma_2), u_1(k-q_1+1), u_2(k-q_1-1)) \frac{(T-\gamma_2)^l}{l!} \tag{30}$$

If $kT+2T \le t < kT+2T+\gamma_1$,
then

$$u_1 = u_1(k-q_1+1),\ u_2 = u_2(k-q_1-1)$$

$$x(kT+2T+\gamma_1) = x(kT+2T) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+2T), u_1(k-q_1+1), u_2(k-q_1-1)) \frac{(\gamma_1)^l}{l!} \tag{31}$$

If $kT+2T+\gamma_1 \le t < kT+2T+\gamma_2$,
then

$$u_1 = u_1(k-q_1+2),\ u_2 = u_2(k-q_1-1)$$

$$x(kT+2T+\gamma_2) = x(kT+2T+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+2T+\gamma_1), u_1(k-q_1+2), u_2(k-q_1-1)) \frac{(\gamma_2-\gamma_1)^l}{l!} \tag{32}$$

If $kT+2T+\gamma_2 \le t < kT+3T$,
then

$$u_1 = u_1(k-q_1+2),\ u_2 = u_2(k-q_1)$$

$$x(kT+3T) = x(kT+2T+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+2T+\gamma_2), u_1(k-q_1+2), u_2(k-q_1)) \frac{(T-\gamma_2)^l}{l!} \tag{33}$$

## 4. Time-Discretization of General Nonlinear Systems with Time-Delay

The general delay-free multi-input nonlinear system in state space form can be written as follows :

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t)) + \sum_{i=1}^{n} g_i(x(t))\, u_i(t) \\ &= f(x(t)) + u_1(t) g_1(x(t)) + u_2(t) g_2(x(t)) + \cdots \\ &\quad + u_n(t) g_n(x(t)) \end{aligned} \tag{34}$$

The values of $A^{[l]}(x, u)$ are evaluated recursively as follows :

$$\begin{aligned} A^{[1]}(x, u) &= f(x) + u_1(t) g_1(x) + u_2(t) g_2(x) + \cdots + u_n(t) g_n(x) \\ A^{[2]}(x, u) &= \dot f(x)\dot x + u_1(t)\dot g_1(x)\dot x + u_2(t)\dot g_2(x)\dot x + \cdots + u_n(t)\dot g_n(x)\dot x \\ &= \frac{\partial A^{[1]}(x, u)}{\partial x}\dot x \\ &\ \ \vdots \\ A^{[l+1]} &= \frac{\partial A^{[l]}(x, u)}{\partial x}(f(x) + u_1(t) g_1(x) + u_2(x) g_2(x) + \cdots + u_n(t) g_n(x)) \end{aligned} \tag{35}$$

The recursive values of Eq. (35) are obtained using the same method as that described in Eq. (7) except the inputs are multiple. From Sec. 3.1, 3.2 and 3.3, the discrete equation can be derived for a nonlinear system having two inputs with time-delay. The results show that Equations (13), (19) and (25) are identical and also the Eqs. (22) and (28) can be derived from Eq. (13) simply by replacing $k$ by $k+1$ because of their location in the second sampling period. Furthermore Eq. (31) can be derived from Eq. (13) by replacing $k$ by $k+2$ since it is located in the third sampling period. Similarly, Eqs. (14), (20) and (26) are identical since the corresponding inputs are in the first sampling period. Equations (14), (23) and (29) are identical since Eqs. (23) and (29) can be derived from Eq. (14) by replacing $k$ by $k+1$ since the inputs are located in the second sampling period and similarly Eq. (32) can be computed from Eq. (14) by replacing $k$ by $k+2$ since it is located in the third sampling period. Also, the Eqs. (15), (21) and (27) are identical and Eqs. (24) and (30) can be obtained from Eq. (15) by replacing $k$ by $k+1$ since the inputs are in the second interval. Finally Eq. (33) can be obtained from Eq. (15) by replacing $k$ by $k+2$ since it is in the third interval. As a result, the inputs and state values can be expressed using the same method no matter where the input is located.

In the 2-input case, the equations can be obtained as follows depending on the interval of time $t$ as shown in Fig. 4.

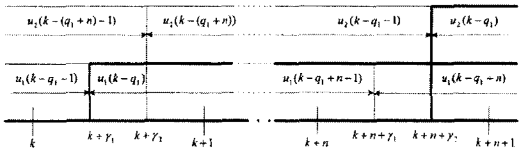

**Fig. 4** Input signal for each time interval in case of two inputs with arbitrary delay

If $kT \le t < kT+\gamma_1$,
then

$$u_1 = u_1(k-q_1-1),\ u_2 = u_2(k-(q_1+n)-1)$$

$$x(kT+\gamma_1) = x(kT) + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_1(k-q_1-1),\ u_2(k-(q_1+n)-1))\frac{(\gamma_1)^l}{l!} \tag{36}$$

If $kT+\gamma_1 \le t < kT+\gamma_2$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-(q_1+n)-1)$$

$$x(kT+\gamma_2) = x(kT+\gamma_1) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_1),\ u_1(k-q_1),\ u_2(k-(q_1+n)-1))\frac{(\gamma_2-\gamma_1)^l}{l!} \tag{37}$$

If $kT+\gamma_2 \le t < kT+T$,
then

$$u_1 = u_1(k-q_1),\ u_2 = u_2(k-(q_1+n))$$

$$x(kT+T) = x(kT+\gamma_2) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_2),\ u_1(k-q_1),\ u_2(k-(q_1+n)))\frac{(T-\gamma_2)^l}{l!} \tag{38}$$

where $k=0, 1, 2, 3, \cdots$ and $q_2 = q_1+n$.

From the above results, the general time-discretization equation of a nonlinear system with delayed multi-input can be derived as follows :

If $kT \le t < kT+\gamma_1$,
then

$$x(kT+\gamma_1) = x(kT) + \sum_{l=1}^{\infty} A^{[l]}(x(kT),\ u_1(k-q_1-1),\ \cdots,\ u_n(k-q_n-1))\frac{(\gamma_1)^l}{l!} \tag{39}$$

If $kT+\gamma_i \le t < kT+\gamma_{i+1}$,
then

$$x(kT+\gamma_{i+1}) = x(kT+\gamma_i) + \sum_{l=1}^{\infty} A^{[l]}\begin{pmatrix} x(kT+\gamma_i),\ u_1(k-q_1),\ \cdots,\ u_i(k-q_i), \\ u_{i+1}(k-q_{i+1}-1),\ \cdots,\ u_n(k-q_n-1) \end{pmatrix}\frac{(\gamma_{i+1}-\gamma_i)^l}{l!} \tag{40}$$

where $1 \le i \le n-1$

$$\vdots$$

If $kT+\gamma_n \le t < kT+T$,
then

$$x(kT+T) = x(kT+\gamma_n) + \sum_{l=1}^{\infty} A^{[l]}(x(kT+\gamma_n),\ u_1(k-q_1),\ \cdots,\ u_{n-1}(k-q_{n-1}),\ u_n(k-q_n))\frac{(T-\gamma_n)^l}{l!} \tag{41}$$

## 5. Simulation

Two examples are considered in the computer simulation to prove the feasibility of the proposed discretization scheme for the delayed multi-input nonlinear system. The examples are a simplified model of maneuvering an automobile (Henk Nijmeijer and Arjan van der Schaft, 1990) and a two-degree of freedom mechanical system. In order to validate the proposed discretization scheme, exact solutions for these systems are also required. In this paper the continuous Matlab ODE solver is used to obtain as an exact solution. In the simulation the discrete values obtained using the Taylor series expansion scheme are compared with those obtained through the continuous Matlab ODE solver for the corresponding sampling period.

### 5.1 Simple second order system

The front axis of a simplified automobile maneuvering system is depicted in Fig. 5. The middle of the axis linking the front wheels has position $(x_1, x_2) \in R^2$, while the rotation of this axis is given by the angle $x_3$. The states $x_1$, $x_2$ related with rolling are directly controlled by the input $u_1$ while the state $x_3$ related with rotation is directly controlled by $u_2$, thus the governing nonlinear differential equation can be expressed as follows :

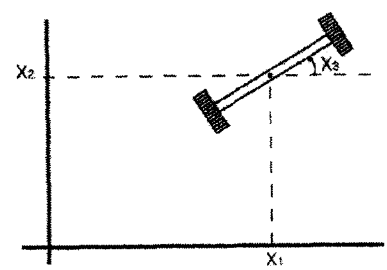

**Fig. 5** Schematic diagram for front axis of automobile

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \sin x_3 \\ \cos x_3 \\ 0 \end{bmatrix} u_1(t-D_1) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_2(t-D_2) \quad (42)$$

Three different sets of inputs will be applied to the system. The first input set contains the delayed inputs $u_1$ and $u_2$ which are in the same sampling period. The other sets are made up of inputs that are not in the same sampling interval. The second set contains inputs whose difference in delays is one sampling period, and the third set contains inputs whose difference in delay is two sampling periods. All of the inputs are assumed to be step functions whose magnitudes are $u_1=1$ and $u_2=2.5$. The simulation results of the three cases are shown in Table 1, Table 2 and Table 3, respectively. The initial conditions are assumed to be $x_1(0)=0$, $x_2(0)=0$, $x_3(0)=30°$ and the sampling period ($T$) is 0.001 sec. The delays of the first case are 0.0005 sec for $u_1$ and 0.0008 sec for $u_2$, thus $\gamma_1$ is 0.0005 sec and $\gamma_2$ is 0.0008 sec. In this case, the input $u_1$ and $u_2$ are located in the same sampling period. The numerical differences between the Matlab solver and the proposed method for state $x_1$ range from $-0.8\times10^{-5}$ to $0.7\times10^{-5}$, while differences for state $x_2$ range from $-0.7\times10^{-5}$ to $0.6\times10^{-5}$ as shown in Table 1. To facilitate the interpretation of the results, the difference between the Taylor and the Matlab results are depicted in Fig. 6. In the second case, the input delays were $D_1=0.0005$ sec and $D_2=0.0018$ sec. The difference between the delay of $u_1$ and $u_2$ is about one sampling period. The numerical differences between the Matlab solver and the proposed method for state $x_1$ range from $-2\times10^{-4}$ to $0.6\times10^{-4}$ and those for state $x_2$ range from $-2\times10^{-4}$ to 0 as shown in Table 2. The differences in the responses the Taylor method and the Matlab solver are shown in Fig. 7. Table 3 shows the simulation results for delay of $D_1=0.0005$ sec and $D_2=0.0028$ sec and the error between the Taylor series and the Matlab solver are depicted in Fig. 8. The difference of each delay is about two sampling periods. The numerical differences between the Matlab solver and the proposed method for state $x_1$ lie in the range from $-0.6\times10^{-4}$ to $1.6\times10^{-4}$ and $0.07\times10^{-4}$ to $2\times10^{-4}$ for state $x_2$.

**Table 1** Time responses of the simplified automobile in the first case

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.1377 | 0.1377 | 0.1414 | 0.1414 |
| 400 | 0.3268 | 0.3268 | 0.1997 | 0.1997 |
| 600 | 0.5208 | 0.5208 | 0.1603 | 0.1602 |
| 800 | 0.6721 | 0.6721 | 0.0326 | 0.0326 |
| 1000 | 0.7436 | 0.7436 | −0.1518 | −0.1518 |
| 1200 | 0.7180 | 0.7180 | −0.3481 | −0.3481 |
| 1400 | 0.6014 | 0.6014 | −0.5080 | −0.5080 |
| 1600 | 0.4224 | 0.4224 | −0.5924 | −0.5924 |
| 1800 | 0.2248 | 0.2248 | −0.5807 | −0.5807 |
| 2000 | 0.0570 | 0.0570 | −0.4757 | −0.4757 |

**Table 2** Time responses of the simplified automobile in the second case

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.1374 | 0.1374 | 0.1417 | 0.1417 |
| 400 | 0.3264 | 0.3263 | 0.2004 | 0.2005 |
| 600 | 0.5204 | 0.5204 | 0.1614 | 0.1615 |
| 800 | 0.6720 | 0.6720 | 0.0341 | 0.0343 |
| 1000 | 0.7440 | 0.7440 | −0.1502 | −0.1499 |
| 1200 | 0.7187 | 0.7188 | −0.3465 | −0.3463 |
| 1400 | 0.6025 | 0.6026 | −0.5067 | −0.5065 |
| 1600 | 0.4236 | 0.4238 | −0.5915 | −0.5914 |
| 1800 | 0.2260 | 0.2262 | −0.5802 | −0.5802 |
| 2000 | 0.0580 | 0.0582 | −0.4756 | −0.4756 |

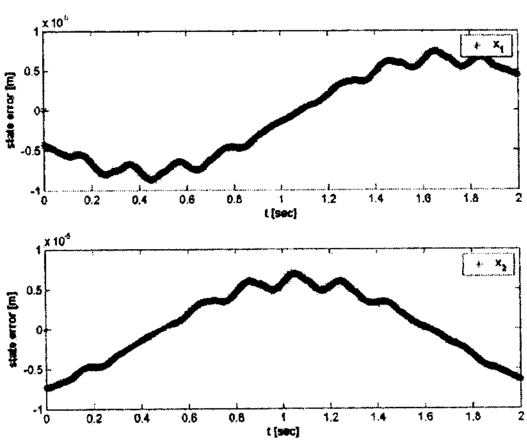


**Fig. 6** State error response of the simplified automobile for the first case

**Table 3** Time responses of the simplified automobile in the third case

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.1370 | 0.1370 | 0.1421 | 0.1421 |
| 400 | 0.3258 | 0.3258 | 0.2014 | 0.2014 |
| 600 | 0.5199 | 0.5200 | 0.1630 | 0.1628 |
| 800 | 0.6719 | 0.6719 | 0.0362 | 0.0360 |
| 1000 | 0.7444 | 0.7444 | −0.1479 | −0.1481 |
| 1200 | 0.7198 | 0.7179 | −0.3443 | −0.3445 |
| 1400 | 0.6040 | 0.6039 | −0.5048 | −0.5050 |
| 1600 | 0.4255 | 0.4253 | −0.5902 | −0.5903 |
| 1800 | 0.2279 | 0.2277 | −0.5795 | −0.5796 |
| 2000 | 0.0595 | 0.0594 | −0.4754 | −0.4754 |

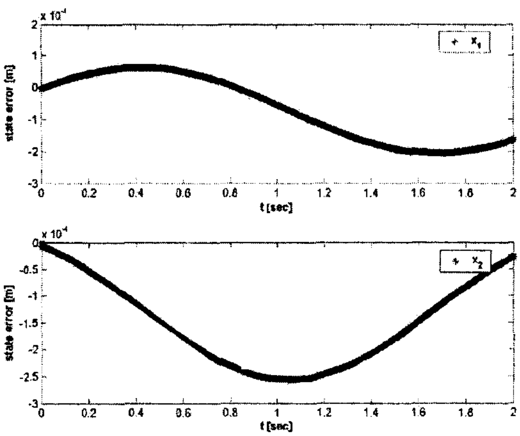


**Fig. 7** State error response of the simplified automobile for the second case

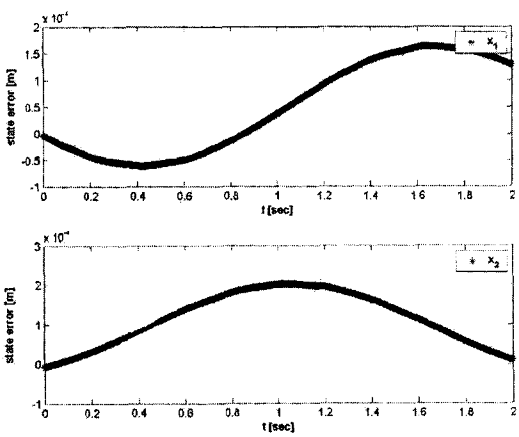


**Fig. 8** State error response of the simplified automobile for the third case

### 5.2 Two Degree-of-Freedom Mechanical System

In this section, a complex two degree-of-freedom mechanical system is examined. The system consists of a slider, spring, damping components, and a pendulum as shown in Fig. 9. The pendulum is hinged to a block mounted on a slider. The slider is free to move along the guides. The motion of the slider is damped by springs, while the rotational resistance in the hinge damps the pendulum. The governing nonlinear differential equations are obtained using Newton's method as follows :

$$
\begin{aligned}
&(m_1+m_2)\ddot{x}+m_1 l(\cos\theta+\mu\sin\theta)\ddot{\theta} \\
&+m_1 l(\mu\cos\theta-\sin\theta)\dot{\theta}^2+\mu(m_1+m_2)g \\
&+2k(x-l_0)=u_1(t-D_1) \\
&m_1 l\cos\theta\ddot{x}+(I_c+m_1 l^2)\ddot{\theta}+m_1 g l\sin\theta \\
&=-M_0\dot{\theta}+u_2(t-D_2)
\end{aligned}
\tag{43}
$$

where all of the parameters are shown in Table 4. The input force $u_1$ is applied to the slider and $u_2$ is applied to the pendulum. Equation (43) can be expressed in the state space form using the following states.

$$
\begin{aligned}
x_1=x,\ x_2=\dot{x},\ x_3=\theta,\ x_4=\dot{\theta} \\
\dot{x}_1=x_2,\ \dot{x}_3=x_4
\end{aligned}
\tag{44}
$$

The state space equation is as follows :

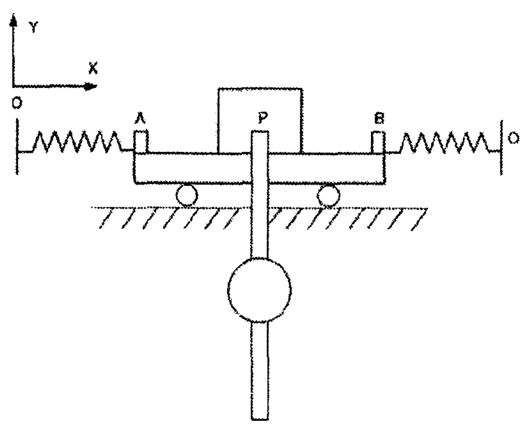


**Fig. 9** Schematic diagram of the 2-DOF mechanical system

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_1+m_2 & 0 & m_1 l(\cos(x_3)+\mu\sin(x_3)) \\ 0 & 0 & 1 & 0 \\ 0 & m_1 l\cos(x_3) & 0 & I_c+m_1 l^2 \end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix} x_2 \\ -m_1 l(\mu\cos(x_3)-\sin(x_3))x_4^2-\mu(m_1+m_2)g-2k\Delta x+u_1 \\ x_4 \\ -M_0x_4+u_2-m_1gl\sin(x_3)\end{bmatrix} \quad (45)$$

The initial conditions are $x(0)=0.3$, $\dot{x}(0)=0$, $\theta(0)=30°$, $\dot{\theta}(0)=0$, and $\triangle x=x-l_0$. This system is simulated with three different sets of inputs as in the previous example. In the first case, the input $u_1$ and $u_2$ are located in the same sampling interval that is the simplest case. The inputs are step functions whose magnitudes are $u_1=5N$ and $u_2=15°$.

It is assumed that the inequality of the delay is $\gamma_1\leq\gamma_2$ for every case. The sampling period $T$ is 0.001 sec. In the first set of delayed inputs, the inputs are within one sampling period, and the delays are assumed to be $\gamma_1=0.0005$ sec, $\gamma_2=0.0008$ sec, and $q_1=0$, $q_2=q_1+0=0$. Therefore the total delay of $u_1$ becomes 0.0005 sec and $u_2$ is 0.0008 sec. These simulation results are shown in Table 5. The response of the proposed method is almost identical to that obtained using the Matlab solver. The numerical difference between the Matlab solver and the Taylor method for state $x_1$ lie in the range $0.8\times10^{-5}$ to $0.8\times10^{-5}$ and from $0.9\times10^{-4}$ to $1.2\times10^{-4}$ for state $x_2$. Those for state $x_3$ range from $7\times10^{-5}$ to $2.8\times10^{-5}$ and from $0.5\times10^{-3}$ to $0.73\times10^{-3}$ for state $x_4$. To facilitate the interpretation of the results, the difference between the Taylor and the Matlab results are depicted in Fig. 10.

**Table 4** System parameters for the 2-DOF mechanical system

| | |
|---|---|
| Mass of slider | $m_1$=0.654 kg |
| Mass of pendulum | $m_2$=0.7925 kg |
| Spring coefficient | $k$=100 N/m |
| Length of the rod | $l$=0:2 m |
| Initial length of the spring | $l_0$=0.025 m |
| Acceleration of the gravity | $g$=9.8 m/s$^2$ |
| Inertia about the center | $I_c$=0.0014 kg·m$^2$ |
| Coefficient of friction | $\mu$=0.2 |
| Dry friction from the pendulum | $M_0$=0.2 kg·m/s$^2$ |

**Table 5** Time responses of the 2-DOF mechanical system for the first case

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.0457 | 0.0457 | −0.0388 | −0.0387 |
| 400 | 0.0272 | 0.0272 | −0.0151 | −0.0152 |
| 600 | 0.0367 | 0.0367 | 0.0294 | 0.0195 |
| 800 | 0.0353 | 0.0353 | −0.0060 | −0.0060 |
| 1000 | 0.0369 | 0.0369 | 0.0043 | 0.0043 |
| 1200 | 0.0356 | 0.0356 | −0.0064 | −0.0065 |
| 1400 | 0.0357 | 0.0357 | 0.0024 | 0.0024 |
| 1600 | 0.0357 | 0.0357 | −0.0008 | −0.0008 |
| 1800 | 0.0358 | 0.0358 | 0.0010 | 0.0010 |
| 2000 | 0.0358 | 0.0358 | −0.0005 | −0.0005 |
| Time step | Matlab ($x_3$) | Taylor ($x_3$) | Matlab ($x_4$) | Taylor ($x_4$) |
| 200 | 0.3468 | 0.3468 | −0.5716 | −0.5717 |
| 400 | 0.2445 | 0.2445 | −0.8124 | −0.8124 |
| 600 | 0.1369 | 0.1369 | 0.0499 | 0.0497 |
| 800 | 0.1922 | 0.1922 | 0.2080 | 0.2081 |
| 1000 | 0.2094 | 0.2094 | 0.0576 | 0.0576 |
| 1200 | 0.2166 | 0.2166 | −0.0246 | −0.0245 |
| 1400 | 0.2058 | 0.2058 | −0.0404 | −0.0404 |
| 1600 | 0.2038 | 0.2038 | 0.0035 | 0.0035 |
| 1800 | 0.2043 | 0.2043 | 0.0054 | 0.0054 |
| 2000 | 0.2059 | 0.2059 | 0.0057 | 0.0057 |

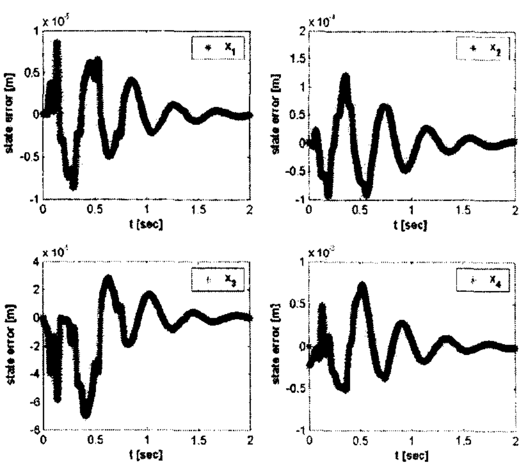


**Fig. 10** State error response of the 2-DOF mechanical system for the first case

In the second case, the delay parameters are assumed to be $\gamma_1=0.0005$ sec, $\gamma_2=0.0008$ sec, $q_1=0$, $q_2=q_1+1=1$ and the difference of the input delays is greater than one sampling period. Therefore, the delay of $u_1$ is 0.0005 sec and the delay of $u_2$ is 0.0018 sec. The simulation results are shown in Table 6. In this case, the differences between the Matlab solver and the proposed method for state $x_1$ lie in the range from $2.1\times10^{-5}$ to $1.8\times10^{-5}$ and from $2.7\times10^{-4}$ to $3\times10^{-4}$ for state $x_2$. Those for $x_3$ range from $2.1\times10^{-4}$ to $0.6\times10^{-4}$ and from $3\times10^{-3}$ to $1.5\times10^{-3}$ for $x_4$. The differences in the responses the Taylor method and the Matlab solver are shown in Fig. 11.

**Table 6** Time responses of the 2-DOF mechanical system for the second test

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.0457 | 0.0457 | −0.0397 | −0.0395 |
| 400 | 0.0272 | 0.0272 | −0.0143 | −0.0145 |
| 600 | 0.0367 | 0.0367 | 0.0290 | 0.0291 |
| 800 | 0.0353 | 0.0353 | −0.0058 | −0.0059 |
| 1000 | 0.0369 | 0.0369 | 0.0042 | 0.0042 |
| 1200 | 0.0356 | 0.0356 | −0.0064 | −0.0064 |
| 1400 | 0.0357 | 0.0357 | 0.0024 | 0.0024 |
| 1600 | 0.0357 | 0.0357 | −0.0008 | −0.0008 |
| 1800 | 0.0358 | 0.0358 | 0.0010 | 0.0010 |
| 2000 | 0.0358 | 0.0358 | −0.0005 | −0.0005 |
| Time step | Matlab ($x_3$) | Taylor ($x_3$) | Matlab ($x_4$) | Taylor ($x_4$) |
| 200 | 0.3462 | 0.3463 | −0.5689 | −0.5692 |
| 400 | 0.2438 | 0.2440 | −0.8123 | −0.8121 |
| 600 | 0.1369 | 0.1369 | 0.0537 | 0.0528 |
| 800 | 0.1924 | 0.1924 | 0.2067 | 0.2070 |
| 1000 | 0.2095 | 0.2095 | 0.0575 | 0.0575 |
| 1200 | 0.2165 | 0.2166 | −0.0250 | −0.0249 |
| 1400 | 0.2058 | 0.2058 | −0.0401 | −0.0402 |
| 1600 | 0.2038 | 0.2038 | 0.0035 | 0.0035 |
| 1800 | 0.2043 | 0.2043 | 0.0054 | 0.0054 |
| 2000 | 0.2059 | 0.2059 | 0.0056 | 0.0056 |

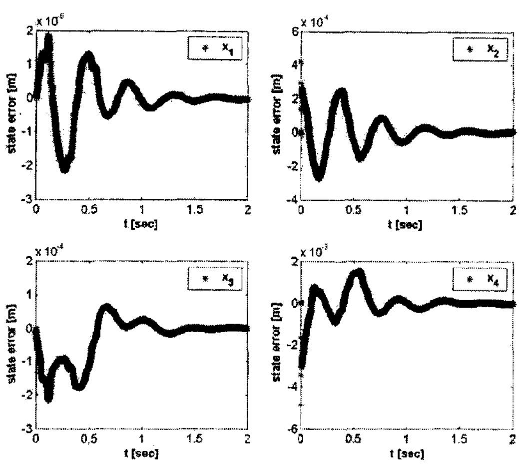


**Fig. 11** State error response of the 2-DOF mechanical system for the second case

The last simulation is for the case where the difference of input delays is more than two sampling periods. In this case, the input delay parameters are $\gamma_1=0.0005$ sec, $\gamma_2=0.0008$ sec, $q_1=0$, $q_2=q_1+2=2$; thus, the total delay of input $u_1$ is 0.005 sec and $u_2$ is 0.0028 sec. The simulation results are shown in Table 7 and the error be-

**Table 7** Time responses of the 2-DOF mechanical system for the third test

| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
|---|---|---|---|---|
| 200 | 0.0457 | 0.0456 | −0.0397 | −0.0397 |
| 400 | 0.0272 | 0.0272 | −0.0143 | −0.0143 |
| 600 | 0.0367 | 0.0367 | 0.0290 | 0.0290 |
| 800 | 0.0353 | 0.0353 | −0.0058 | −0.0058 |
| 1000 | 0.0369 | 0.0369 | 0.0042 | 0.0042 |
| 1200 | 0.0356 | 0.0356 | −0.0064 | −0.0064 |
| 1400 | 0.0357 | 0.0357 | 0.0024 | 0.0024 |
| 1600 | 0.0357 | 0.0357 | −0.0008 | −0.0008 |
| 1800 | 0.0358 | 0.0358 | 0.0010 | 0.0010 |
| 2000 | 0.0358 | 0.0358 | −0.0005 | −0.0005 |
| Time step | Matlab ($x_1$) | Taylor ($x_1$) | Matlab ($x_2$) | Taylor ($x_2$) |
| 200 | 0.3462 | 0.3458 | −0.5689 | −0.5668 |
| 400 | 0.2438 | 0.2434 | −0.8123 | −0.8119 |
| 600 | 0.1369 | 0.1370 | 0.0536 | 0.0559 |
| 800 | 0.1924 | 0.1925 | 0.2067 | 0.2059 |
| 1000 | 0.2095 | 0.2095 | 0.0575 | 0.0574 |
| 1200 | 0.2165 | 0.2165 | −0.0250 | −0.0252 |
| 1400 | 0.2058 | 0.2058 | −0.0401 | −0.0400 |
| 1600 | 0.2038 | 0.2038 | 0.0035 | 0.0035 |
| 1800 | 0.2043 | 0.2043 | 0.0054 | 0.0054 |
| 2000 | 0.2059 | 0.2059 | 0.0056 | 0.0056 |

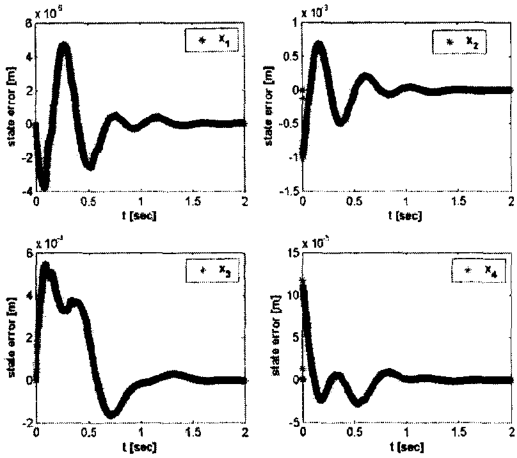

**Fig. 12** State error response of the 2-DOF mechanical system for the third case

tween the Taylor series and the Matlab solver are depicted in Fig. 12. The error ranges for state $x_1$ are from $3.7\times10^{-5}$ to $4.6\times10^{-5}$ and from $-1\times10^{-3}$ to $0.6\times10^{-3}$ for state $x_2$; error for state $x_3$ ranges from $-1.6\times10^{-4}$ to $5.4\times10^{-4}$; error for state $x_4$ ranges from $2.7\times10^{-3}$ to $11.7\times10^{-3}$.

As a result, it is shown that the proposed Taylor-series expansion scheme discretizes a nonlinear system with time delayed multi-input accurately.

## 6. Conclusions

This study proposes a new approach to the discrete-time representation of nonlinear control systems with delayed multi-input. It is based on the ZOH assumption and the Taylor-series expansion. In this paper a discrete-time system is derived directly from the continuous-time system without undergoing any transformation, and the proposed scheme explicitly accounts for the presence of time-delay. The resulting time-discretization provides a finite-dimensional representation of nonlinear control systems with delayed multi-input, allowing for the possible application of existing nonlinear controller design techniques. The proposed discretization algorithm are tested using some case studies with increasing complexity, demonstrating their satisfactory convergence characteristics. The extension to nonlinear system with time-varying delay and to systems with state and /or output delay is feasible and it will be the subject of future research.

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